RYUTHP-95/4
August, 1995

A Geometrical Formulation of the Renormalization Group Method for Global Analysis II: Partial Differential Equations

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Abstract

It is shown that the renormalization group (RG) method for global analysis can be formulated in the context of the classical theory of envelopes: Several examples from partial differential equations are analyzed. The amplitude equations which are usually derived by the reductive perturbation theory are shown to be naturally derived as the equations describing the envelopes of the local solutions obtained in the perturbation theory.

1 Introduction

Renormalization group (RG) equations[1] appear in various fields of science. In quantum field theory[2], the RG equation improves results obtained in the perturbation theory. In recent years, the improvement of the effective potential[2, 3, 4] has acquired a renewed interest[5]. The RG equation has also a remarkable success in statistical physics especially in the critical phenomena [2]. One may also note that there is another successful theory of the critical phenomena called coherent anomaly method (CAM)[6]; CAM utilizes *envelopes* of susceptibilities and other thermodynamical quantities as a function of temperature. It is well known that Feigenbaum [7] applied RG equation to deduce a universality of some chaotic phenomena.

Recently, Illinois group[8, 9] have shown that the RG equation can be also used for non-quantum mechanical problems: They proposed to use the RG equation to get an asymptotic behavior of solutions of differential equations including ones of singular and reductive perturbation problems in a unified way. Mathematically, the RG equation is used to improve the global behavior of the local solutions which were obtained in the perturbation theory. This fact suggests that the RG method may be formulated in a purely mathematical way

without recourse to the notion of the RG. In the previous paper[11], we showed that the RG method can be formulated in the context of the classical theory of envelopes[12]: We pointed out that the RG equation is nothing but the envelope equation, and gave a proof why the RG equation can give a globally improved solution to ordinary differential equations(ODE's). In fact, if a family of the curves $\{C_\tau\}_\tau$ in the x-y plane is represented by the equation $F(x, y, \tau) = 0$, the equation $G(x, y) = 0$ representing the envelope E is given by eliminating the parameter τ from the equation

$$\left.\frac{\partial F(x, y, \tau)}{\partial \tau}\right|_{\tau=x} = 0. \tag{1.1}$$

Here we have chosen the parameter to be the x-coordinate of the point of the tangency of a curve C_τ and the envelope E. The relevance of the envelope equation Eq.(1.1) and the (Gell-Mann-Low type) RG equations is evident; the parameter τ corresponds to the renormalization point. Thus one would also readily recognize that improving the effective potential in quantum field theory is constructing the envelope of the effective potentials with the renormalization point varied.[11]

The purpose of the present paper is to show that the formulation can be naturally extended to a class of partial differential equations (PDE's); the PDE's dealt in the present work include a dissipative nonlinear hyperbolic equation, one- and two-dimensional Swift-Hohenberg equations, damped Kuramoto-Shivashinsky equation[14] and Barlenblatt equation[13]. Most of these examples were examined by the Illinois group [8, 9, 10]. Therefore there will be some overlaps in the exposition with theirs. It is inevitable, however, because our purpose is to give a purely mathematical formulation of their method without recourse to the notion of the renormalization group. We will show that the amplitude equations usually obtained by the reductive perturbation methods are given naturally as envelope equations. In the course of the formulation, relevance of the characteristic manifold as the branch stripmanifold will be indicated to constructing the global solutions. In Appendix A, we give a short summary of the classical theory of envelope surfaces; for PDE's, envelope *surfaces* may be more relevant in some cases than envelope *curves* discussed in [11]. In Appendix B, we examine Barenblatt's equation in our approach; for this equation, only the anomalous exponent of the long-time behavior of the solution is given.

2 Dissipative nonlinear hyperbolic equation

We first consider the following slightly dissipative nonlinear hyperbolic equation[9] to apply the envelope theory[1] to construct a global solution:

$$\frac{\partial u}{\partial t} + \lambda(u)\frac{\partial u}{\partial x} = \eta\frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

[1]See Appendix A and [11] for classical theory of envelopes.

where $\lambda(u)$ is a sufficiently smooth function of u, and η is a positive constant. Following [9], we consider a small amplitude wave in the background of constant solution u_0;

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots. \tag{2.2}$$

Here ϵ denotes the amplitude of the wave. $\lambda(u)$ is expanded as

$$\lambda(u) = \lambda_0 + \epsilon\lambda'(u_0)u_1 + \cdots, \tag{2.3}$$

where $\lambda_0 \equiv \lambda(u_0)$. We further assume that the dissipation is weak in the sense $\eta \sim \epsilon$; we write as $\eta = \mu\epsilon$, where $\mu = O(1)$. Then equating the coefficients of ϵ^n $(n = 0, 1, 2, ...)$, we have

$$\begin{aligned} \partial_t u_1 + \lambda_0 \partial_x u_1 &= 0, \\ \partial_t u_2 + \lambda_0 \partial_x u_2 &= -\lambda_0' u_1 \partial_x u_1 + \mu \partial_x^2 u_1, \end{aligned} \tag{2.4}$$

and so on. Here, $\lambda_0' = \lambda'(u_0)$.

It is now convenient to introduce the new variable $\xi \equiv x - \lambda_0 t$, the characteristic direction of the unperturbed equation; we describe the solutions in terms of (ξ, t). Then

$$\begin{aligned} \partial_t u_1 &= 0, \\ \partial_t u_2 &= -\lambda_0' u_1 \partial_\xi u_1 + \mu \partial_\xi^2 u_1, \end{aligned} \tag{2.5}$$

and so on. One readily gets

$$\begin{aligned} u_1(\xi, t) &= F_0(\xi), \\ u_2(\xi, t) &= (t - t_0)(-\lambda_0' F_0(\xi)\partial_\xi F_0(\xi) + \mu \partial_\xi^2 F_0(\xi)), \end{aligned} \tag{2.6}$$

where $F_0(\xi)$ is an arbitrary function of ξ. We note that u_2 is a secular term.[2]

We have thus an approximate solution to Eq. (2.1)

$$u(\xi, t; t_0) = u_0 + \epsilon F_0(\xi) + \epsilon^2 (t - t_0)(-\lambda_0' F_0(\xi)\partial_\xi F_0(\xi) + \mu\partial_\xi^2 F_0(\xi)) + O(\epsilon^3), \tag{2.7}$$

where we have made it explicit that u is dependent on an arbitrary time t_0. We stress that this (approximate) solution is only valid for t around t_0.

Now, geometrically speaking, we have a family of surfaces S_{t_0} represented by $u(\xi, t; t_0)$ with a parameter t_0. Let us obtain the envelope surface E of this family of surfaces, following the classical theory of envelopes given in Appendix A: we represent E by $u_{_E}(\xi, t)$.

We first note that $F_0(\xi)$ may be functionally dependent on the arbitrary initial time t_0:

$$u(\xi, t; t_0) = u_0 + \epsilon F_0(\xi, t_0) + \epsilon^2 (t - t_0)(-\lambda_0' F_0(\xi, t_0)\partial_\xi F_0(\xi, t_0) + \mu \partial_\xi^2 F_0(\xi, t_0)) + O(\epsilon^3). \tag{2.8}$$

[2]One might have added another arbitrary function $G_0(\xi)$of ξ to u_2. However, the effect of G_0 could be renormalized away to F_0.

It is natural to set the tangent curve C_{t_0} of E and S_{t_0} to lie along $t = t_0$-line, which is parallel to the characteristic direction (ξ-direction):

$$\mathrm{C}_{t_0}: \quad t = t_0, \quad u = u(\xi, t_0, t_0), \quad \xi = \xi. \tag{2.9}$$

Then the envelope equation reads

$$\frac{\partial u}{\partial t_0} = 0, \quad \text{and} \;\; t_0 = t, \tag{2.10}$$

which leads to

$$\partial_t F(\xi, t) + \epsilon \lambda'(u_0) F \partial_\xi F = \eta \partial_\xi^2 F. \tag{2.11}$$

This is Burgers equation which is usually obtained in the reductive perturbation theory. Then

$$u_{_E}(\xi, t) = u(\xi, t; t) = u_0 + \epsilon F(\xi, t). \tag{2.12}$$

Now one may wonder if $u_{_E}(\xi, t)$ satisfies Eq. (2.1), although $u(\xi, t; t_0)$ does up to $O(\epsilon^3)$ for any t_0. We remark that the question is not trivial because $u_{_E}(\xi, t) \equiv u(\xi, t; t)$. We shall show that the answer is yes. In fact, the time derivative of $u_{_E}(\xi, t)$ at $t = \forall t_0$ coincides with $\partial u(\xi, t; t_0)/\partial t$ at $t = t_0$;

$$\begin{aligned} \left.\frac{\partial u_{_E}(\xi, t)}{\partial t}\right|_{t=t_0} &= \left.\frac{\partial u(\xi, t; t_0)}{\partial t}\right|_{t=t_0} + \left.\frac{\partial u(\xi, t_0; t_0')}{\partial t_0'}\right|_{t_0'=t_0}, \\ &= \left.\frac{\partial u(\xi, t; t_0)}{\partial t}\right|_{t=t_0}, \end{aligned} \tag{2.13}$$

where Eq. (2.10) has been used. Furthermore, needless to say, $u_{_E}(\xi, t_0) = u(\xi, t_0, t_0)$. Thus, at $t = \forall t_0$,

$$\frac{\partial u_{_E}}{\partial t} + \lambda(u_E) \frac{\partial u_{_E}}{\partial x} = \eta \frac{\partial^2 u_{_E}}{\partial x^2} + O(\epsilon^3). \tag{2.14}$$

Thus one sees that our envelope function $u_{_E}$ satisfies the Eq.(2.1) uniformly for t in the global range.

A comment is in order here: one may say that to get the approximate but global solution $u_{_E}$ from the local solution Eq.(2.8), we have utilized the fact that the uniqness of the solution of the Cauchy problem is violated when the initial data are given along the characteristic manifold (curve); see Appendix A for characteristic manifolds(curves).

3 One-dimensional Swift-Hohenberg equation

In this section, we deal with the one-dimensional Swift-Hohenberg equation[15, 14]:

$$\hat{L}_1 u = \epsilon u - u^3, \qquad \hat{L}_1 \equiv \partial_t + (\partial_x^2 + k^2)^2, \tag{3.1}$$

where ϵ is a small parameter. We shall show that the envelope of local solutions of Eq. (3.1) satisfies the time-dependent Ginzburg-Landau equation.

Following [8], we scale u as

$$u = \sqrt{\epsilon}\phi. \tag{3.2}$$

Then ϕ satisfies

$$\hat{L}_1\phi = \epsilon(\phi - \phi^3). \tag{3.3}$$

We solve this equation in the perturbation theory, expanding ϕ as $\phi = \phi_0 + \epsilon\phi_1 + ...$; we have

$$\hat{L}_1\phi_0 = 0, \quad \hat{L}_1\phi_1 = \phi_0 - \phi_0^3, \tag{3.4}$$

and so on. As the 0-th order solution, we take

$$\phi_0(x,t) = Ae^{ikx} + \text{c.c.}, \tag{3.5}$$

where c.c. denotes the complex conjugate. Then the 1st-order equation reads

$$\hat{L}_1\phi_1 = \mathcal{A}e^{ikx} - A^3e^{3ikx} + \text{c.c.}, \tag{3.6}$$

where

$$\mathcal{A} \equiv A(1 - 3|A|^2). \tag{3.7}$$

A special solution to Eq. (3.6) is found to be[9]

$$\phi_1 = \mathcal{A}\{\mu_1(t - t_0) - \frac{\mu_2}{8k^2}(x^2 - x_0^2)\}e^{ikx} + \frac{A^3}{64k^3}e^{3ikx} + \text{c.c.}, \tag{3.8}$$

with $\mu_1 + \mu_2 = 1$. We note that the secular terms have appeared in ϕ_1 because of the first term in r.h.s. of Eq. (3.6).

Now one may say that we have a family of surfaces $S_{t_0x_0}$ represented by

$$\begin{aligned}\phi(x,t;x_0,t_0) \quad = \quad & [Ae^{ikx} + \epsilon\mathcal{A}\{\mu_1(t - t_0) - \frac{\mu_2}{8k^2}(x^2 - x_0^2)\}e^{ikx} + \epsilon\frac{A^3}{64k^3}e^{3ikx}] \\ & +\text{c.c.},\end{aligned} \tag{3.9}$$

parametrized by t_0 and x_0. Let us obtain the envelope of the family of the surfaces in two steps, by assuming that the amplitude A is dependent on x_0 and t_0. First, fixing x_0, we obtain the envelope E_1 of the surfaces with t_0 being the parameter. The resultant envelope E_{x_0} has the parameter x_0; then we obtain the envelope of the family of the surfaces E_1.

The first step is achieved by setting

$$\frac{\partial\phi}{\partial t_0} = 0, \quad \text{with} \quad t_0 = t. \tag{3.10}$$

This is the condition to get the envelope that has the common tangent curve with $S_{x_0t_0}$ along x-direction. Thus we have the equation for $A(x_0, t)$;

$$\partial_t A = \mu_1 \epsilon \mathcal{A} + O(\epsilon^2). \tag{3.11}$$

Then the envelope E_1 is represented by

$$\begin{aligned}\phi_{E_1}(x, t; x_0) &\equiv \phi(x, t; x_0, t_0 = t), \\ &= [A(x_0, t)e^{ikx} - \epsilon \mathcal{A}(x_0, t)\frac{\mu_2}{8k^2}(x^2 - x_0^2)e^{ikx} + \epsilon \frac{A(x_0, t)^3}{64k^3} e^{3ikx}] \\ &\quad + \text{c.c.}.\end{aligned} \tag{3.12}$$

The envelope E of the family of the surfaces given by Eq. (3.12) is obtained as follows;

$$\frac{\partial \phi_{E_1}}{\partial x_0} = 0, \quad \text{with} \quad x_0 = x, \tag{3.13}$$

which leads to

$$\partial_x A(x, t) = -\mu_2 \epsilon \mathcal{A} \frac{x}{4k^2} + O(\epsilon^2). \tag{3.14}$$

Here we have utilized the fact that $\partial_x A$ is $O(\epsilon)$. Differentiating this equation with respect to x, we get

$$\partial_x^2 A(x, t) = -\epsilon \frac{\mu_2}{4k^2} \mathcal{A} + O(\epsilon^2). \tag{3.15}$$

Here we note again that $\partial_x A$ is $O(\epsilon)$.

Now combining Eq.'s (3.10) and (3.14), we have

$$\partial_t A = 4k^2 \partial_x^2 A + \epsilon A(1 - 3|A|^2), \tag{3.16}$$

up to $O(\epsilon^2)$. The envelope is represented by

$$\begin{aligned}\phi_E(x, t) &\equiv \phi_{E_1}(x, t; x_0 = x), \\ &= \left[A(x, t)e^{ikx} + \epsilon \frac{A(x, t)^3}{64k^4} e^{3ikx}\right] + \text{c.c.}.\end{aligned} \tag{3.17}$$

A comment is in order here: To derive Eq.(3.15), we started from the first-order differential equation Eq.(3.13) with respect to x_0. One could start from the second-order differential equation as is done in [9];

$$\left.\frac{\partial^2 \phi}{\partial x_0^2}\right|_{x_0 = x} = 0, \tag{3.18}$$

which gives Eq.(3.15) as an exact relation without the remainder of $O(\epsilon^2)$. We remark that the geometrical meaning of Eq.(3.18) is not clear in contrast to Eq.(3.13). In our formulation, Eq.(3.18) is derived as an approximate relation, as is shown below.

Now one may ask the question as to the relation of $\phi_E(x,t)$ given in Eq. (3.17) and the original equation Eq.(3.1): The answer is that $\phi_E(x,t)$ satisfies Eq.(3.1) up to $O(\epsilon^2)$ uniformly in the global domain due to the very envelope conditions Eq.(3.10) and Eq.(3.13). In fact, for $\forall t = t_0$,

$$\frac{\partial \phi_E}{\partial t}\bigg|_{t=t_0} = \frac{\partial \phi(t,t_0)}{\partial t}\bigg|_{t=t_0} + \frac{\partial \phi(t,t_0)}{\partial t_0}\bigg|_{t=t_0} = \frac{\partial \phi(t,t_0)}{\partial t}\bigg|_{t=t_0}, \tag{3.19}$$

on account of Eq.(3.10). Similary, for $\forall x = x_0$, one can easily verify that

$$\frac{\partial^2 \phi_E}{\partial x^2}\bigg|_{x=x_0} = \frac{\partial^2 \phi}{\partial x^2}\bigg|_{x=x_0} + O(\epsilon^2), \tag{3.20}$$

on account of Eq.'s (3.12), (3.13) and (3.18). For instance, $\partial^2\phi_E/\partial x\partial x_0|_{x=x_0} = O(\epsilon^2)$.

4 Damped Kuramoto-Shivashinsky equation

In this section, we deal with the (one-dimensional) damped Kuramoto-Shivashinsky equation, which is given by

$$\hat{L}_1 u = \epsilon u - u\partial_x u, \tag{4.1}$$

where $\hat{L}_1$ is defined in Eq.(3.1). This equation is not examined by the Illinois group.

By scaling u as

$$u = \sqrt{\epsilon}\phi, \tag{4.2}$$

we have[3]

$$\hat{L}_1 \phi = -\epsilon^{1/2}\phi\partial_x\phi + \epsilon\phi. \tag{4.3}$$

We first try to solve this equation by the perturbation theory with the expansion

$$\phi = \phi_0 + \epsilon^{1/2}\phi_1 + \epsilon\phi_1 + ... \tag{4.4}$$

The equations for $\phi_0, \phi_1, \phi_2 ...$ are found to be

$$\hat{L}_1\phi_0 = 0, \quad \hat{L}_1\phi_1 = -\phi_0\partial_x\phi_0, \quad \hat{L}_1\phi_2 = \phi_0 - (\phi_0\partial_x\phi_1 + \phi_1\partial_x\phi_0), \tag{4.5}$$

and so on.

As the 0-th order solution, we take

$$\phi_0 = Ae^{ikx} + \text{c.c}, \tag{4.6}$$

[3]If one scales u as $u = \epsilon\phi$, one will get a different equation from that given below.

where A is a constant. Then ϕ_1 is found to be

$$\phi_1 = -\frac{i}{9k^2}A^2 e^{2ikx} + \text{c.c.}. \tag{4.7}$$

Thus the equation for ϕ_2 reads

$$\hat{L}_1\phi_2 = A(1 - \frac{|A|^2}{9k^2})e^{ikx} - \frac{A^3}{3k^3}e^{3ikx} + \text{c.c.}, \tag{4.8}$$

which is in a similar form with Eq.(3.6). One easily gets for a special solution to this equation

$$\phi_2 = A(1 - \frac{|A|^2}{9k^2})\{\mu_1(t-t_0) - \frac{\mu_2}{8k^2}(x^2 - x_0^2)\}e^{ikx} - \frac{A^3}{192k^6}e^{3ikx} + \text{c.c.}. \tag{4.9}$$

Thus we reach the solution in the perturbation theory up to $O(\epsilon^{3/2})$,

$$\begin{aligned}\phi(x,t;x_0,t_0) \quad &= \quad \Big[A\mathrm{e}^{ikx} - i\epsilon^{1/2}\frac{A^2}{9k^3}e^{2ikx} + \epsilon A(1 - \frac{|A|^2}{9k^2})\{\mu_1(t-t_0) - \frac{\mu_2}{8k^2}(x^2 - x_0^2)\}e^{ikx} \\ &\qquad -\epsilon\frac{A^3}{192k^6}e^{3ikx}\Big] + \text{c.c.}.\end{aligned} \tag{4.10}$$

Now we have a family of surfaces $S_{x_0t_0}$ in x-t-ϕ plane represented by $\phi(x,t;x_0,t_0)$ with x_0 and t_0 being the parameters. We repeat the procedure of the previous section to obtain the envelope E of the family of the surfaces: We first note that A may depend on x_0 and t_0, i.e., $A = A(x_0,t_0)$. Then fixing x_0, we first obtain the envelope of $S_{x_0t_0}$ with t_0 being the parameters. The envelope E_1 is obtained by

$$\frac{\partial\phi}{\partial t_0} = 0, \quad \text{with} \quad t_0 = t, \tag{4.11}$$

where we have assumed that the tangent curve (the characteristic curve) is along x-direction by setting $t_0 = t$. The above equation leads to

$$\partial_t A(x_0,t) = \epsilon\mu_1 A(1 - \frac{|A|^2}{9k^2}) + O(\epsilon^{3/2}). \tag{4.12}$$

With A satisfying this equation, the envelope E_1 is represented by

$$\begin{aligned}\phi_{E_1}(x,t;x_0) \quad &\equiv \quad \phi(x_0,t_0;x_0,t_0=t) \\ &= \quad \Big[A\mathrm{e}^{ikx} - i\epsilon^{1/2}\frac{A^2}{9k^3}e^{2ikx} - \epsilon A(1 - \frac{|A|^2}{9k^2})\frac{\mu_2}{8k^2}(x^2 - x_0^2)e^{ikx} \\ &\qquad -\epsilon\frac{A^3}{192k^6}e^{3ikx}\Big] + \text{c.c.}.\end{aligned} \tag{4.13}$$

This function can be regarded as representing a family of surfaces with x_0 being the parameter. The envelope E of this family of surfaces are obtained by setting

$$\frac{\partial\phi_{E_1}}{\partial x_0} = 0, \quad \text{with} \quad x_0 = x, \tag{4.14}$$

which leads to

$$\partial_x A(x,t) = -\epsilon\frac{\mu_2}{4k^2}A(1-\frac{|A|^2}{9k^2})x + O(\epsilon^{3/2}). \tag{4.15}$$

Here we have utilized the fact that $\partial_x A \sim O(\epsilon)$. Further differentiating with respect to x, one has

$$\partial_x^2 A = -\epsilon\frac{\mu_2}{4k^2}A(1-\frac{|A|^2}{9k^2}) + O(\epsilon^{3/2}). \tag{4.16}$$

Combining Eq.'s (4.12) and (4.16), one sees that the amplitude satisfies equation

$$(\partial_t - 4k^2\partial_x^2)A = \epsilon A(1-\frac{|A|^2}{9k^2}), \tag{4.17}$$

up to $O(\epsilon^2)$.

With this amplitude, the envelope E is given by

$$\begin{aligned}\phi_E(x,t) &\equiv \phi(x,t;x_0=x), \\ &= Ae^{ikx} - i\epsilon^{1/2}\frac{A^2}{9k^2}e^{2ikx} - \epsilon\frac{A^3}{192k^6}e^{3ikx} + \text{c.c.}.\end{aligned} \tag{4.18}$$

A couple of comments are in order here:

(1) Eq.(4.16) could be obtained as an exact relation by imposing the condition that

$$\left.\frac{\partial^2\phi}{\partial x_0^2}\right|_{x_0=x} = 0. \tag{4.19}$$

We must note, however, that the geometrical meaning of this condition is not clear. In contrast, in our formulation, the second derivative is evaluated to be

$$\left.\frac{\partial^2\phi}{\partial x_0^2}\right|_{x_0=x} = O(\epsilon^2). \tag{4.20}$$

(2) As was done in the preceding section, one can easily show that $\phi_E(x,t)$ satisfies the original equation Eq.(4.1) up to $O(\epsilon^2)$ but uniformly in the global domain.

5 Two-dimensional Swift-Hohenberg equation

In this section, we deal with the two-dimensional Swift-Hohenberg equation given by

$$\hat{L}_2 u = \epsilon u - u^3, \qquad \hat{L}_2 = \partial_t + (\partial_x^2 + \partial_y^2 + k^2)^2. \tag{5.1}$$

Scaling u as

$$u = \sqrt{\epsilon}\phi, \tag{5.2}$$

we have

$$\hat{L}_2\phi = \epsilon(\phi - \phi^3). \tag{5.3}$$

We first solve this equation in the naive perturbation theory expanding $\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + ...$;

$$\hat{L}_2\phi_0 = 0, \qquad \hat{L}_2\phi_n = \phi_{n-1} - \phi_{n-1}^3, \quad (n = 1, 2, ...). \tag{5.4}$$

If we assume the roll solution along y-axis for the zero-th order equation,

$$\phi_0 = Ae^{ikx} + \text{c.c.}, \tag{5.5}$$

then the solution up to $O(\epsilon^2)$ is found to be [10]

$$\begin{aligned}\phi(x, t; x_0, t_0) &= \Big[A + \epsilon\{\mu_1(t - t_0) - \mu_2\frac{x^2 - x_0^2}{8k^2} + \mu_3\frac{xy^2 - x_0y_0^2}{8ik} + \mu_4\frac{y^4 - y_0^4}{4!}\}\mathcal{A}\Big]e^{ikx} \\ &\quad +\text{c.c.}, \quad \text{with} \quad \mathcal{A} \equiv A(1 - 3|A|^2),\end{aligned} \tag{5.6}$$

where $\sum_{i=1\sim 4}\mu_i = 1$ and x_0, y_0 and t_0 are arbitrary constants. Here we have omitted the terms that do not give rise to secular terms.

Now one may regard that we have a family of "surfaces" in the x-y-t-ϕ space with x_0, y_0 and t_0 being the parameters. Let us obtain the envelope of this "surfaces" in the three steps by noting that A may be dependent on x_0, y_0 and t_0: First we regard that only t_0 is the parameter of the "surfaces" with both x_0 and y_0 fixed. The condition reads

$$\frac{\partial\phi}{\partial t_0} = 0, \qquad \mathrm{t}_0 = \mathrm{t}. \tag{5.7}$$

This condition may be also regarded as the one for the envelope curve of a family of curves in t-ϕ plane with x and y being fixed. The condition leads to

$$\partial_t A(x_0, y_0, t) = \epsilon\mu_1\mathcal{A} + O(\epsilon^2). \tag{5.8}$$

Inserting this solution, we have the envelope

$$\begin{aligned}\phi_{E_1}(x, y, t; x_0, y_0) &\equiv \phi(x, y, t; x_0, y_0, t_0 = t) \\ &= \Big[A(x_0, y_0, t) + \epsilon\{-\mu_2\frac{x^2 - x_0^2}{8k^2} + \mu_3\frac{xy^2 - x_0y_0^2}{8ik} + \mu_4\frac{y^4 - y_0^4}{4!}\}\mathcal{A}\Big]e^{ikx} \\ &\quad +\text{c.c.},\end{aligned} \tag{5.9}$$

which we may regard as a family of curves in x-ϕ plane with x_0 being the parameter where y and y_0 are fixed. The envelope E_2 for this family of curves is given by setting

$$\frac{\partial\phi_{E_1}}{\partial x_0} = 0, \qquad \text{with} \quad x_0 = x, \tag{5.10}$$

which leads to

$$\partial_x A(x, y_0, t) = -\epsilon(\frac{\mu_2}{4k^2}x - \frac{\mu_3}{8ik}y_0^2)\mathcal{A} + (\epsilon^2). \tag{5.11}$$

Accordingly, the envelope is represented by

$$\begin{aligned}\phi_{E_2}(x, y, t; y_0) &\equiv \phi_{E_1}(x, y, t; x_0 = x, y_0) \\ &= \Big[A(x, y_0, t)e^{ikx} + \epsilon\{\mu_3 x\frac{y^2 - y_0^2}{8ik} + \mu_4\frac{y^4 - y_0^4}{4!}\}\mathcal{A}e^{ikx}\Big] + \text{c.c.}, \end{aligned} \tag{5.12}$$

which is regarded as representing a family of curves in y-ϕ plane. The envelope of ϕ_{E_2} is obtained as usual by setting

$$\frac{\partial \phi_{E_2}}{\partial y_0} = 0, \quad \text{with} \quad y_0 = y, \tag{5.13}$$

which leads to

$$\partial_y A(x, y, t) = \epsilon(\frac{\mu_3}{4ik}xy + \frac{\mu_4}{3!}y^3)\mathcal{A} + O(\epsilon^2). \tag{5.14}$$

Differentiating Eq.'s (5.11) and (5.14), we have

$$\partial_x^2 A(x, y, t) = -\epsilon\frac{\mu_2}{4k^2}\mathcal{A} + O(\epsilon^2), \tag{5.15}$$

$$\partial_x\partial_y^2 A(x, y, t) = -\epsilon\frac{\mu_3}{4ik}\mathcal{A} + O(\epsilon^2), \tag{5.16}$$

$$\partial_y^4 A(x, y, t) = \epsilon\mu_4\mathcal{A} + O(\epsilon^2), \tag{5.17}$$

where we have utilized the fact that

$$\partial_x A \sim O(\epsilon), \quad \partial_y A \sim O(\epsilon). \tag{5.18}$$

Combining these equation together with Eq.(5.8), we finally reach

$$\partial_t A - 4k^2\partial_x^2 A + 4ik\partial_x\partial_y^2 A + \partial_y^4 A = \epsilon A(1 - 3|A|^2), \tag{5.19}$$

up to $O(\epsilon^2)$.

With this amplitude, the envelope is given by

$$\begin{aligned}\phi_E(x, y, t) &= \phi_{E_1}(x, y, t; y_0 = y), \\ &= A(x, y, t)e^{ikx} + \text{c.c.}.\end{aligned} \tag{5.20}$$

A few comments are in order here:

(1) By imposing that the any order of the differentiation of ϕ with respect to x_0 and y_0 should vanish, one could get Eq.'s (5.15-17) as exact relations [10], although the geometrical meaning of these conditions are unclear. In our formulation, instead, the following are derived as approximate relations

$$\frac{\partial^2\phi_{E1}}{\partial^2 x_0}\bigg|_{x_0=x} = O(\epsilon^2), \quad \frac{\partial^3\phi_{E2}}{\partial x\partial^2 y_0}\bigg|_{y_0=y} = O(\epsilon^2), \quad \frac{\partial^4\phi_{E2}}{\partial^4 y_0}\bigg|_{y_0=y} = O(\epsilon^2). \tag{5.21}$$

(2) It can be shown that ϕ_E satisfies Eq.(5.1) up to $O(\epsilon^2)$ but uniformly in the global domain owing to the above relations together with Eq. (5.8).

6 A brief summary and concluding remarks

We have shown in the present paper that the RG method for global analysis can be formulated for partial differential equations, too. An interesting equation which is treated by the Illinois group[8] but left untouched in the text is Barenblatt's equation. A complete global solution to this equation is not given in the RG method[8] but only the anomalous exponent for the long-time behavior is obtained. For completeness, we show that the same result for the anomalous exponent can be obatined in our envelope theory in Appendix B. Thus together with our previous paper[11] where ordinary differential equations are discussed, we have shown that almost all the classes of problems treated in the RG method by the Illinois group[8, 9, 10] are nicely formulated on the basis of the classical theory of envelopes without recourse to the notion of the renormalization group. Actually, this is natural because the resulting equations indeed describe the amplitudes of the nonlinear waves as given by the solutions of the nonlinear equations.

It is already indicated[8, 9] that the renormalizability is equivalent to the solvability of equations[16]. It would be interesting to apply the theory developed here to systems of equations[4] and see possible geometrical meaning of the solvability condition.

In deriving the whole envelopes of local solutions, we have taken a multi-step approach in sections 3, 4 and 5. It is interesting that a kind of multi-step approach is also proposed for improving the effective potentials in quantum field theories with multi-scales; see the paper by Ford[5]. It may imply that our approach that identifies the RG equation with the envelope equation naturally leads to the effective potentials as given by Ford when applied to quantum field theories.[18]

Acknowledgements

The author acknowledges G. C. Paquette, who gave lectures on the RG method at Ryukoku University in June 1994. He is also grateful to R. Kobayashi, who organized the seminar where these lectures were given. He thanks Y. Morita, the conversation with whom motivated the author to think about the mathematical structure of the RG method seriously. The author is grateful to M. Yamaguti for his interest in this work, encouragement and useful comments on Cauchy's problem and Burgers equation. He also acknowledges Y. Oono for his correspondence and sending the preprint[10] prior to publication. The author thanks J. Matsukidaira for discussions on applications of the envelope theory to systems of ODE's. A part of the present work was completed while the author stayed at Brookhaven Natinal Laboratory(BNL) as a summer-program visitor in August 3-14, 1995. He thanks BNL and especially R. Pisarski for their hospitality.

[4] The RG method or the envelope theory developed in [11] and here is of course applicable to systems of ODE's[17].

Appendix A

In this appendix, we give a short review of the classical theory of envelopes[12]. We shall first consider envelopes in three-dimensional space. i.e., envelope surfaces. Then the extension to higher dimensional cases will be briefly described.[5]

Let $\{S_\tau\}_\tau$ be a family of surfaces parametrized by τ in the x-y-u space; here S_τ is represented by the equation

$$\Phi(x, y, u; \tau) = 0. \tag{A.1}$$

We suppose that $\{S_\tau\}_\tau$ has the envelope E, which is represented by the equation

$$\Psi(x, y, u) = 0. \tag{A.2}$$

The problem is to obtain $\Psi(x, y, u)$ from $\Phi(x, y, u; \tau)$.

Let E and S_τ has a common tangent plane on a common curve C_τ; with τ being varied, $\{C_\tau\}_\tau$ forms E. C_τ is called a *characteristic curve.* The necessary condition for S_τ to have an envelope E is given as follows. Let C_τ be represented by $x = x(\sigma, \tau),\ y = y(\sigma, \tau),\ u = u(\sigma, \tau)$, with σ being a parameter. We assume that

$$\mathrm{rank}\begin{pmatrix} x_\sigma & y_\sigma & u_\sigma \\ x_\tau & y_\tau & u_\tau \end{pmatrix} = 2, \tag{A.3}$$

in order for $\{C_\tau\}_\tau$ to give a non-degenerate surface E. Now, since C_τ is on S_τ, $\Phi(x(\sigma, \tau), y(\sigma, \tau), u(\sigma, \tau); \tau) = 0$. Differentiating this equation with respect to τ, one has

$$\Phi_x x_\tau + \Phi_y y_\tau + \Phi_u u_\tau + \Phi_\tau = 0, \tag{A.4}$$

where $\Phi_x \equiv \partial\Phi/\partial x$ and so on. On the other hand, since $(x_\tau, y_\tau, u_\tau) \equiv \mathbf{t}_\tau$ is a tangent vector both of E and S_τ, and $(\Phi_x, \Phi_y, \Phi_u) \equiv \mathbf{n}_\tau$ is a normal vector of S_τ,

$$\mathbf{n}_\tau \cdot \mathbf{t}_\tau = \Phi_x x_\tau + \Phi_y y_\tau + \Phi_u u_\tau = 0. \tag{A.5}$$

Combining Eq.'s (A.4) and (A.5), one has

$$\Phi_\tau \equiv \frac{\partial\Phi(x, y, u; \tau)}{\partial\tau} = 0, \tag{A.6}$$

as a necessary condition for $\{S_\tau\}$ to have an envelope E. Thus, solving Eq. (A.6) for $\tau = \tau(x, y, u)$, one finally gets

$$\Psi(x, y, u) = \Phi(x, y, u; \tau(x, y, u)) = 0. \tag{A.7}$$

Accordingly, the characteristic curve is given by the conditions;

$$\Psi(x, y, u) = 0 \quad \text{and} \quad \tau(x, y, u) = \text{const.} \tag{A.8}$$

[5]In [11], a review is given on how to construct envelope curves.

Note that the second equation gives a constraint on x, y and u.

We remark that the sufficient conditions for the existence of an envelope is supplemented by [12]

$$\Phi_{\tau\tau} = 0. \tag{A.9}$$

When S_τ is given by $u = \varphi(x, y; \tau)$, the condition Eq. (A.6) is reduced to

$$\frac{\partial \varphi(x, y; \tau)}{\partial \tau} = 0, \tag{A.10}$$

which gives τ as a function of x and y. Thus we get for the envelope

$$u = \varphi_{_E}(x, y) = \varphi(x, y; \tau(x, y)). \tag{A.11}$$

The characteristic curve is accordingly given by

$$u = \varphi_{_E}(x, y)\Big|_{\tau(x,y)=\text{const.}}. \tag{A.12}$$

As an example, let

$$u = \varphi(x, y, \tau) \equiv \exp(\tau y)(1 + y(x - \tau)) + \exp(-x). \tag{A.13}$$

Then the equation $\partial\varphi/\partial\tau = 0$ gives

$$\tau = x, \tag{A.14}$$

except at $y \neq 0$. Thus the envelope is given by

$$u = \exp(xy) + \exp(-x). \tag{A.15}$$

The characteristic curve C_τ is given by

$$x = \tau = \text{constant} \equiv x_0, \quad \text{and} \quad u = \exp(x_0\ y) + \exp(-x_0). \tag{A.16}$$

We remark that $\partial^2\varphi/\partial\tau^2 \neq 0$ provided that $y \neq 0$.

The theory can be extended to the envelope of a family of hyper-surfaces $\{S_\tau\}_\tau$ in the $(n+1)$-dimensional space $\mathbf{R}^{n+1}$. Let $\{S_\tau\}_\tau$ be represented by the equation $\Phi(x_1, x_2, \cdots, x_n, u; \tau) = 0$ with a parameter set $\tau = (\tau_1, \tau_2, \cdots, \tau_{n-1})$. Then the function representing the envelope is given by eliminating parameters τ_i $(i = 1, 2, \cdots, n-1)$ from

$$\Phi(x_1, x_2, \cdots, x_n, u; \tau) = 0, \quad \frac{\partial \Phi}{\partial \tau_i} = 0, \quad (i = 1, 2, \cdots, n-1). \tag{A.17}$$

A few comments are in order here:

(i) The envelope of a family of surfaces has usually an improved global nature compared with the surfaces in the family. So it is natural that the theory of envelopes may have some power for global analysis.

(ii) It should be stressed here that Eq.(A.6), which we call the envelope equation, has the same form as renormalization group (RG) equations. Rather, it should be said conversely; RG equations in general are envelope equations[11]. This is the reason why RG equations "improve" things; one should note that the improvement by an RG equation usually means that a function with a better global nature is constructed from functions with a local nature only valid around the renormalization point.

(iii) Let $u = \varphi(x, y; \sigma, \tau)$ be a complete solution of a PDE

$$F(x, y, u, p, q) = 0, \qquad (p \equiv \partial u/\partial x, \quad q \equiv \partial u/\partial y), \tag{A.18}$$

where σ and τ are constant. Assuming a functional dependence of σ on τ, i.e., $\sigma = A(\tau)$, one can obtain the envelope as follows;

$$\frac{du}{d\tau} = \frac{dA}{d\tau}\frac{\partial u}{\partial \sigma} + \frac{\partial u}{\partial \tau} = 0, \tag{A.19}$$

from which τ is given as a function of x and y, namely, $\tau = \tau(x, y)$. Then one sees that the envelope function $u = \varphi_{_E}(x, y; [A]) = \varphi(x, y, A(\tau(x, y)), \tau(x, y))$ is the general solution to $F(x, y, u, p, q) = 0$.

Appendix B

In this Appendix, we examine a non-linear partial differential equation which has an anomalous exponent in the long-time behavior of its solutions;

$$(\partial_t - \frac{1}{2}\partial_x^2)u(t, x) = \theta(-\partial_t u)\partial_x^2 u. \tag{B.1}$$

This is called Barlenblatt's equation.[13, 8] This equation is a nonlinear PDE with the step function in r.h.s. Without this term ($\epsilon = 0$), the equation is a simple diffusion equation and has a self-similar form after a long time; $u(x, t) \sim t^{-1/2} f(x^2/t)$. However, with $\epsilon \neq 0$, the long-time behavior is found to be

$$u(x, t) \sim t^{-(\frac{1}{2}+\alpha)} F(\frac{x^2}{t}). \tag{B.2}$$

The anomalous exponent α can be interpreted as an anomalous dimension [8]. The appearance of the anomalous dimension is due to the fact that the scale l with a spatial dimension which characterizes the initial distribution of $u(x, 0)$ can not be neglected even in the long-time limit $t \to \infty$ in contrast to the case of the simple diffusion equation [13]. We are interested in determining the anomalous dimension α in the perturbation theory.

To solve the problem, we shall follow [8] for a while. First we convert the equation to an integral equation;

$$\begin{aligned} u(x, t) \;&=\; \int_{-\infty}^{\infty} dy G(x - y, t) u(y, 0) \\ &\quad + \frac{\epsilon}{2}\int_0^t ds \int_{-\infty}^{\infty} dy G(x - y, t - s)\theta[-\partial_s u(y, s)]\partial_y^2 u, \end{aligned} \tag{B.3}$$

where

$$G(x,t) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}} \tag{B.4}$$

is the Green's function.[6] We expand u as

$$u(x,t) = u_0(x,t) + \epsilon u_1(x,t) + \cdots. \tag{B.5}$$

As an initial condition, we take

$$u(x,0) = u_0(x,0) = \frac{m_0}{\sqrt{2\pi l^2}} \mathrm{e}^{-\frac{x^2}{2l^2}}, \tag{B.6}$$

where m_0 and l are parameters. Then one finds[8] that

$$u(x,t) = \frac{m_0}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}} \{1 - \frac{\epsilon}{\sqrt{2\pi e}} \ln \frac{t}{l^2}\} + O(\epsilon^2) + O(l^2/t, \epsilon), \tag{B.7}$$

where we have retained in u_1 only the term which may contribute to the anomalous dimension.

We renormalize m_0 at $t = t_0$ as follows;

$$m(t_0) = Z(t_0, l) m_0, \quad Z(t_0, l) = 1 - \frac{\epsilon}{\sqrt{2\pi e}} \ln \frac{t_0}{l^2}, \tag{B.8}$$

accordingly,

$$u(0, t_0) = m(t_0)/\sqrt{2\pi t_0}, \tag{B.9}$$

and then

$$u(x,t;t_0) = \frac{m}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}} \{1 - \frac{\epsilon}{\sqrt{2\pi e}} \ln \frac{t}{t_0}\} + O(\epsilon^2) + O(l^2/t, \epsilon). \tag{B.10}$$

Here we have made the t_0 dependence of u explicit.

We now leave the line of argument of Goldenfeld et al[8]. We have a family of surfaces represented by $u(x,t;t_0)$; the surfaces are parametrized by t_0. Let us obtain the envelope E of this family of surfaces. We can suppose that the tangent curve C_{t_0} (the characteristic curve) is along $t = t_0$:

$$t = t_0, \quad u = u(x, t_0; t_0). \tag{B.11}$$

[6] If the Green's function is defined as usual by

$$(\partial_t - \frac{1}{2}\partial_x^2)\mathcal{G}(x,t) = \delta(x)\delta(t),$$

$$\mathcal{G}(x,t) = \theta(t) G(x,t),$$

where $\theta(t)$ is the step function.

Now according to the theory given in Appendix A, the envelope is given by the following condition;

$$\frac{\partial u}{\partial t_0} = 0 \quad \text{with} \quad t_0 = t. \tag{B.12}$$

This equation is reduced to an equation for m,

$$\dot{m} + \alpha \frac{m}{t} = 0, \quad \alpha = \epsilon / \sqrt{2\pi e}, \tag{B.13}$$

hence

$$m(t) = \bar{m} t^{-\alpha} \tag{B.14}$$

with $\bar{m}$ being a constant number. Thus the envelope is given by

$$u_{_E}(x,t) = u(x,t;t) = \bar{m} t^{-(1/2+\alpha)} F(x^2/t), \tag{B.15}$$

hence the anomalous exponent reads $\alpha = \epsilon/\sqrt{2\pi e}$, which of course coincides with the result of Goldenfeld et al.[7]

[7]It should be mentioned that $u_{_E}(x,t)$ thus obtained does *not* satisfy the Barlenblatt's equation even in the order of ϵ; this is because the Gaussian form for $F(x^2/t)$ is not correct. To obtain the precise form of $F(x^2/t)$, one may insert $u_{_E}(x,t)$ into the Barlenblatt's equation and solve $F(\xi)$.

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